On Strongly Regular Graphs, the Friendship Theorem, Lovász Function, and Shannon Capacity of Graphs

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Graph Spectrum

Throughout this presentation,

- $\mathsf{G} = (\mathsf{V}(\mathsf{G}), \mathsf{E}(\mathsf{G}))$ is a finite, undirected, and simple graph of order $|\mathsf{V}(\mathsf{G})| = n$ and size $|\mathsf{E}(\mathsf{G})| = m$.
- $\mathbf{A} = \mathbf{A}(\mathsf{G})$ is the *adjacency matrix* of the graph.
- The eigenvalues of $\mathbf{A}$ are given in decreasing order by

$$\lambda_{\max}(\mathsf{G}) = \lambda_1(\mathsf{G}) \geq \lambda_2(\mathsf{G}) \geq \ldots \geq \lambda_n(\mathsf{G}) = \lambda_{\min}(\mathsf{G}). \tag{1.1}$$

- The *spectrum* of G is a multiset that consists of all the eigenvalues of $\mathbf{A}$, including their multiplicities.

Orthogonal Representation of Graphs

Definition 1.1

Let G be a finite, undirected and simple graph.
An orthogonal representation of G in $\mathbb{R}^d$

$$i \in \mathsf{V}(\mathsf{G}) \;\mapsto\; \mathbf{u}_i \in \mathbb{R}^d$$

such that

$$\mathbf{u}_i^{\mathrm{T}} \mathbf{u}_j = 0, \quad \forall \{i, j\} \notin \mathsf{E}(\mathsf{G}).$$

An orthonormal representation of G: $\|\mathbf{u}_i\| = 1$ for all $i \in \mathsf{V}(\mathsf{G})$.

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In an orthogonal representation of a graph G:

- non-adjacent vertices: mapped to orthogonal vectors;
- adjacent vertices: not necessarily mapped to non-orthogonal vectors.

Lovász ϑ-function

Let G be a finite, undirected and simple graph.

The Lovász ϑ-function of G is defined as

$$\vartheta(\mathsf{G}) \triangleq \min_{\mathbf{u},\mathbf{c}} \max_{i \in \mathsf{V}(\mathsf{G})} \frac{1}{\left(\mathbf{c}^{\mathrm{T}}\mathbf{u}_i\right)^2}, \tag{1.2}$$

where the minimum is taken over

- all orthonormal representations $\{\mathbf{u}_i : i \in \mathsf{V}(\mathsf{G})\}$ of G, and
- all unit vectors $\mathbf{c}$.

The unit vector $\mathbf{c}$ is called the *handle* of the orthonormal representation.

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$$\left|\mathbf{c}^{\mathrm{T}} \mathbf{u}_i\right| \leq \|\mathbf{c}\| \, \|\mathbf{u}_i\| = 1 \implies \vartheta(\mathsf{G}) \geq 1,$$

with equality if and only if G is a complete graph.

An Orthonormal Representation of a Pentagon

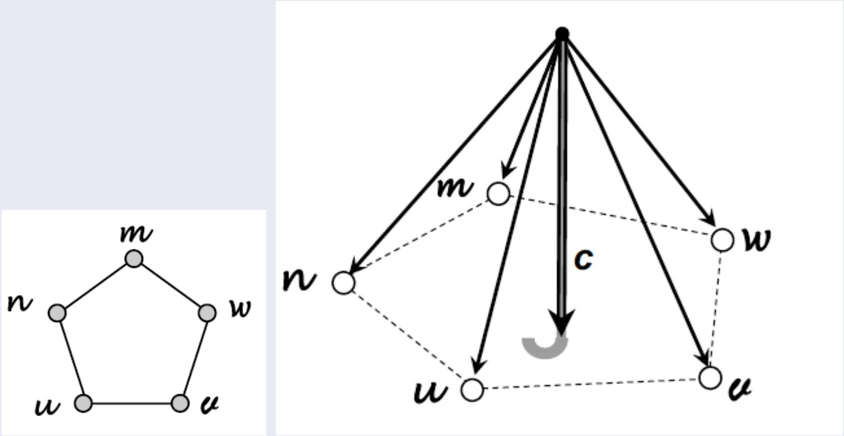

Figure 1: A 5-cycle graph and its orthonormal representation (also known as Lovász umbrella). Calculation shows that $\vartheta(\mathrm{C}_5) = \sqrt{5}$ (Lovász, 1979).

Lovász ϑ-function (Cont.)

- $\mathbf{A}$ is the $n \times n$ adjacency matrix of G $\quad (n \triangleq |\mathsf{V}(\mathsf{G})|)$;
- $\mathbf{J}_n$ is the all-ones $n \times n$ matrix;
- $\mathcal{S}_+^n$ is the set of all $n \times n$ positive semidefinite matrices.

Semidefinite program (SDP), with strong duality, for computing $\vartheta(\mathsf{G})$:

$$
\begin{array}{l}
\text{maximize } \ \text{Trace}(\mathbf{B}\,\mathbf{J}_n) \\
\text{subject to} \\
\begin{cases}
\mathbf{B} \in \mathcal{S}_+^n, \ \ \text{Trace}(\mathbf{B}) = 1, \\
A_{i,j} = 1 \ \Rightarrow \ B_{i,j} = 0, \quad i, j \in [n].
\end{cases}
\end{array}
$$

Computational complexity: $\exists$ algorithm (based on the ellipsoid method) that numerically computes $\vartheta(\mathsf{G})$, for every graph G, with precision of r decimal digits, and polynomial-time in n and r.

Lovász ϑ-function (Cont.)

Let $\alpha(\mathsf{G})$, $\omega(\mathsf{G})$, and $\chi(\mathsf{G})$ denote the independence number, clique number, and chromatic number of a graph G. Then,

1. Sandwich theorem:

$$\alpha(\mathsf{G}) \leq \vartheta(\mathsf{G}) \leq \chi(\overline{\mathsf{G}}), \tag{1.3}$$

$$\omega(\mathsf{G}) \leq \vartheta(\overline{\mathsf{G}}) \leq \chi(\mathsf{G}). \tag{1.4}$$

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2. Computational complexity:
   - $\alpha(\mathsf{G})$, $\omega(\mathsf{G})$, and $\chi(\mathsf{G})$ are NP-hard problems.
   - However, the numerical computation of $\vartheta(\mathsf{G})$ is in general feasible by convex optimization (SDP problem).

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   - However, the numerical computation of $\vartheta(\mathsf{G})$ is in general feasible by convex optimization (SDP problem).

3. Hoffman-Lovász inequality: Let G be d-regular of order n. Then,
$$\vartheta(\mathsf{G}) \leq -\frac{n\,\lambda_n(\mathsf{G})}{d - \lambda_n(\mathsf{G})}, \tag{1.5}$$
with equality if G is edge-transitive.

Strongly Regular Graphs

Let G be a d-regular graph of order n. It is a *strongly regular* graph (SRG) if there exist nonnegative integers λ and μ such that

- Every pair of adjacent vertices have exactly λ common neighbors;
- Every pair of distinct and non-adjacent vertices have exactly μ common neighbors.

Such a strongly regular graph is said to belong to the family $\text{srg}(n, d, \lambda, \mu)$.

Theorem: Adjacency Spectrum of Strongly Regular Graphs

The following spectral properties of strongly regular graphs hold:

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The following spectral properties of strongly regular graphs hold:

- A strongly regular graph has at most three distinct eigenvalues.
- Let G be a connected strongly regular graph in the family $\text{srg}(n, d, \lambda, \mu)$ (i.e., $\mu > 0$). Then, its adjacency spectrum consists of three distinct eigenvalues, where the largest eigenvalue is given by $\lambda_1(\mathsf{G}) = d$ with multiplicity 1, and the other two distinct eigenvalues of its adjacency matrix are given by

$$p_{1,2} = \tfrac{1}{2}\left(\lambda - \mu \pm \sqrt{(\lambda-\mu)^2 + 4(d-\mu)}\right), \tag{1.6}$$

with the respective multiplicities

$$m_{1,2} = \tfrac{1}{2}\left(n - 1 \mp \frac{2d + (n-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2 + 4(d-\mu)}}\right). \tag{1.7}$$

Theorem (cont.)

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- A disconnected strongly regular graph is a disjoint union of m identical complete graphs K_r, where $m \geq 2$ and $r \in \mathbb{N}$. It belongs to the family $\text{srg}(mr, r-1, r-2, 0)$, and its adjacency spectrum is $\{(r-1)^{[m]}, (-1)^{[m(r-1)]}\}$, where superscripts indicate the multiplicities of the eigenvalues, thus having two distinct eigenvalues.

Theorem 1.2 (Bounds on Lovász function of Regular Graphs, I.S., '23)

Let G be a d-regular graph of order n, which is a non-complete and non-empty graph. Then, the following bounds hold for the Lovász ϑ-function of G and its complement $\overline{\mathsf{G}}$:

1)

$$\frac{n - d + \lambda_2(\mathsf{G})}{1 + \lambda_2(\mathsf{G})} \leq \vartheta(\mathsf{G}) \leq -\frac{n\lambda_n(\mathsf{G})}{d - \lambda_n(\mathsf{G})}. \tag{1.8}$$

- Equality holds in the leftmost inequality if $\overline{\mathsf{G}}$ is both vertex-transitive and edge-transitive, or if G is a strongly regular graph;
- Equality holds in the rightmost inequality if G is edge-transitive, or if G is a strongly regular graph.

Cont. of Theorem 1.2

2)

$$1 - \frac{d}{\lambda_n(\mathsf{G})} \leq \vartheta(\overline{\mathsf{G}}) \leq \frac{n\big(1+\lambda_2(\mathsf{G})\big)}{n-d+\lambda_2(\mathsf{G})}. \tag{1.9}$$

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A Common Sufficient Condition

All inequalities hold with equality if G is strongly regular. (Recall that the graph G is strongly regular if and only if $\overline{\mathsf{G}}$ is so).

Lovász Function of Strongly Regular Graphs (I.S., '23)

Let G be a strongly regular graph in the family srg(n, d, λ, μ). Then,

$$\vartheta(\mathsf{G}) = \frac{n\,(t+\mu-\lambda)}{2d+t+\mu-\lambda}, \tag{1.10}$$

$$\vartheta(\overline{\mathsf{G}}) = 1 + \frac{2d}{t+\mu-\lambda}, \tag{1.11}$$

where

$$t \triangleq \sqrt{(\mu-\lambda)^2 + 4(d-\mu)}. \tag{1.12}$$

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New Relation for Strongly Regular Graphs

$$\vartheta(\mathsf{G})\,\vartheta(\overline{\mathsf{G}}) = n, \tag{1.13}$$

holding not only for all vertex-transitive graphs (Lovász '79), but also for all strongly regular graphs (that are not necessarily vertex-transitive).

In general, we have $\vartheta(\mathsf{G})\,\vartheta(\overline{\mathsf{G}}) \geq n$ (Lovász, 1979).

Corollary 1.3 (Lovász ϑ-Function of SRGs (I.S., '23))

The Lovász ϑ-function of strongly regular graphs (SRGs) is uniquely determined by its four parameters (n, d, λ, μ).

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Example: Chang Graphs

- Chang graphs are three non-isomorphic strongly regular graphs with parameters srg$(28, 12, 6, 4)$.
- These graphs are not vertex-transitive and also not edge-transitive.
- The clique numbers of these 3 graphs are 5, 6, 6.

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- These graphs are not vertex-transitive and also not edge-transitive.
- The clique numbers of these 3 graphs are 5, 6, 6.
- Nevertheless, they have the same Lovász ϑ-function, being equal to 4. The Lovász ϑ-function of the complements, all srg$(28, 15, 6, 10)$, is 7.
- Note that, indeed, $\vartheta(\mathsf{G})\,\vartheta(\overline{\mathsf{G}}) = 28$ for these three graphs, although they are not vertex-transitive (but SRGs).

Question

Strongly regular graphs that belong to the same family srg(n, d, λ, μ), where $\mu > 0$, are connected and cospectral graphs. Although these graphs are not necessarily isomorphic, are there any pairs of connected and cospectral graphs with distinct Lovász ϑ-functions?

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The next result gives a negative answer to this question.

Theorem 1.4 (A result with an explicit construction (I.S, 2024))

For every even integer $n \geq 14$, it is constructively proven that there exist connected, irregular, cospectral, and nonisomorphic graphs on n vertices, being jointly cospectral with respect to their adjacency, Laplacian, signless Laplacian, and normalized Laplacian matrices, while also sharing identical independence, clique, and chromatic numbers, but being distinguished by their Lovász ϑ-functions.

Theorem 1.5 (Second-Largest and Least Eigenvalues (I.S., '23))

Let G be a d-regular graph of order n, which is non-complete and non-empty. Then,

$$\lambda_n(\mathsf{G}) \leq -\frac{d\,(n-d+\lambda_2(\mathsf{G}))}{d+(n-1)\,\lambda_2(\mathsf{G})}, \tag{1.14}$$

or equivalently,

$$\lambda_2(\mathsf{G}) \geq -\frac{d\,(n-d+\lambda_n(\mathsf{G}))}{d+(n-1)\,\lambda_n(\mathsf{G})}. \tag{1.15}$$

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These inequalities hold with equality if and only if G is strongly regular.

1. From our earlier bounds, it follows that the inequality holds with equality if G is a strongly regular graph.
2. We prove that if G is regular, then equality holds if and only if G is strongly regular.

We next provide an original proof of the following celebrated theorem by Erdös, Rényi and Sós (1966), based on our expression for the Lovász ϑ-function of strongly regular graphs (and their complements, which are also strongly regular graphs).

Theorem 1.6 (Friendship Theorem)

Let G be a finite graph in which any two distinct vertices have a single common neighbor. Then, G has a vertex that is adjacent to every other vertex.

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Let G be a finite graph in which any two distinct vertices have a single common neighbor. Then, G has a vertex that is adjacent to every other vertex.

A Human Interpretation of Theorem 1.6

- There is a party with n people, where every two people have precisely one common friend in that party.
- Theorem 1.6 asserts that one of these people is everybody's friend.
- Indeed, construct a graph whose vertices represent the n people, and every two vertices are adjacent if and only if they represent two friends. The claim then follows from Theorem 1.6.

Remark 1 (On the Friendship Theorem - Theorem 1.6)

- The windmill graph (see Figure 2) has the desired property, and it turns out to be the only one graph with that property.
- The friendship theorem does not hold for infinite graphs.

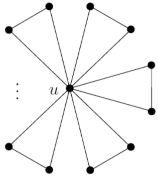

Figure 2: Windmill graph.

Alternative Proof of Theorem 1.6 (I.S., 25)

Suppose the assertion is false, and G is a counterexample — a finite graph in which any two distinct vertices have a single common neighbor, yet no vertex in G is adjacent to all other vertices. A contradiction is obtained by the following proof outline:

- It is shown that the graph is regular.
- It is then shown that the graph is strongly regular srg$(n, k, 1, 1)$.
- If $k = 0$ or $k = 2$, then $\mathsf{G} = \mathsf{K}_1$ or $\mathsf{G} = \mathsf{K}_3$, respectively, which satisfy the assertion of the theorem. Hence, next assume that $k \geq 3$.
- By the theorem hypothesis, it follows that $\omega(\mathsf{G}) = \chi(\mathsf{G}) = 3$.
- By the sandwich theorem $\omega(\mathsf{G}) \leq \vartheta(\overline{\mathsf{G}}) \leq \chi(\mathsf{G})$, so $\vartheta(\overline{\mathsf{G}}) = 3$.
- Based on the expression for the Lovász ϑ-function $\vartheta(\overline{\mathsf{G}}) = 1 + \frac{k}{\sqrt{k-1}}$.
- This leads to a contradiction for all $k \geq 3$.

The sandwich theorem for the Lovász ϑ-function applied to strongly regular graphs gives the following result.

Corollary 1.7 (Bounds on Parameters of SRGs)

Let G be a strongly regular graph in the family $\text{srg}(n, d, \lambda, \mu)$. Then,

$$\alpha(\mathsf{G}) \leq \left\lfloor \frac{n\,(t+\mu-\lambda)}{2d+t+\mu-\lambda} \right\rfloor \tag{1.16}$$

$$\omega(\mathsf{G}) \leq 1 + \left\lfloor \frac{2d}{t+\mu-\lambda} \right\rfloor, \tag{1.17}$$

$$\chi(\mathsf{G}) \geq 1 + \left\lceil \frac{2d}{t+\mu-\lambda} \right\rceil, \tag{1.18}$$

$$\chi(\overline{\mathsf{G}}) \geq \left\lceil \frac{n\,(t+\mu-\lambda)}{2d+t+\mu-\lambda} \right\rceil, \tag{1.19}$$

with

$$t \triangleq \sqrt{(\mu-\lambda)^2 + 4(d-\mu)}. \tag{1.20}$$

Examples: Bounds on Parameters of SRGs

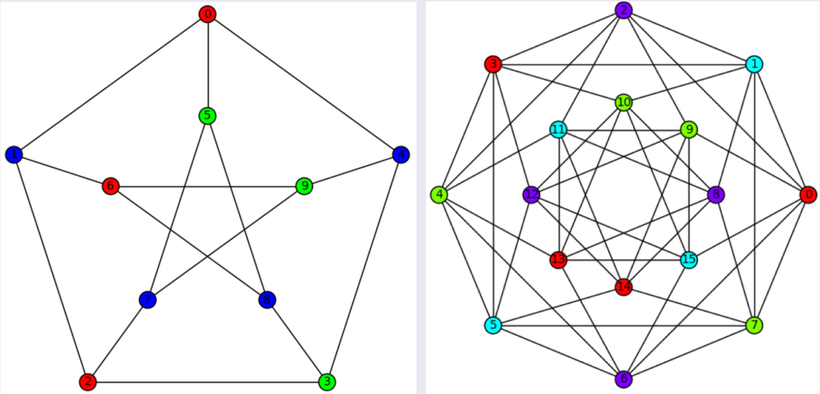

Figure 3: The Petersen graph is srg$(10, 3, 0, 1)$ (left), and the Shrikhande graph is srg$(16, 6, 2, 2)$ (right). Their chromatic numbers are 3 and 4, respectively.

Schläfli Graph

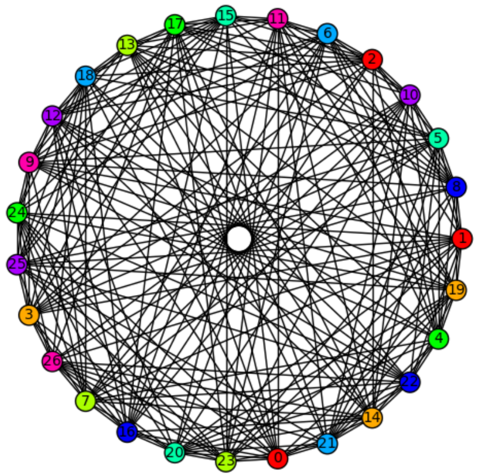

Figure 4: Schläfli graph is $\mathrm{srg}(27, 16, 10, 8)$ with chromatic number $\chi(\mathsf{G}) = 9$.

Examples: Bounds on Parameters of SRGs (Cont.)

1. Let G_1 be the Petersen graph. Then, the bounds on the independence, clique, and chromatic numbers of G are tight:

$$\alpha(\mathsf{G}_1) = 4, \quad \omega(\mathsf{G}_1) = 2, \quad \chi(\mathsf{G}_1) = 3. \tag{1.21}$$

2. The bounds on the chromatic numbers of the Schläfli graph (G_2), Shrikhande graph (G_3) and Hall-Janko graph (G_4) are tight:

$$\chi(\mathsf{G}_2) = 9, \quad \chi(\mathsf{G}_3) = 4, \quad \chi(\mathsf{G}_4) = 10. \tag{1.22}$$

3. For the Shrikhande graph (G_3),
   - the bound on its independence number is also tight: $\alpha(\mathsf{G}_3) = 4$,
   - its upper bound on its clique number is, however, not tight (it is equal to 4, and $\omega(\mathsf{G}_3) = 3$).

Strong Product of Graphs

Let G and H be two graphs. The strong product $\mathsf{G} \boxtimes \mathsf{H}$ is a graph with

- vertex set: $\mathsf{V}(\mathsf{G} \boxtimes \mathsf{H}) = \mathsf{V}(\mathsf{G}) \times \mathsf{V}(\mathsf{H})$,
- two distinct vertices (g, h) and (g', h') in $\mathsf{G} \boxtimes \mathsf{H}$ are adjacent if the following two conditions hold:
  1. $g = g'$ or $\{g, g'\} \in \mathsf{E}(\mathsf{G})$,
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Strong products are commutative and associative.

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Strong products are commutative and associative.

Strong Powers of Graphs

Let

$$\mathsf{G}^{\boxtimes k} \triangleq \underbrace{\mathsf{G} \boxtimes \ldots \boxtimes \mathsf{G}}_{\mathsf{G} \text{ appears } k \text{ times}}, \quad k \in \mathbb{N} \tag{1.23}$$

denote the k-fold strong power of a graph G.

Properties of the Lovász ϑ-Function with Strong Products

1. Factorization for strong product graphs: For all graphs G and H,

$$\vartheta(\mathsf{G} \boxtimes \mathsf{H}) = \vartheta(\mathsf{G})\, \vartheta(\mathsf{H}), \tag{1.24}$$

$$\vartheta(\overline{\mathsf{G} \boxtimes \mathsf{H}}) = \vartheta(\overline{\mathsf{G}})\, \vartheta(\overline{\mathsf{H}}). \tag{1.25}$$

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2. The equality

$$\sup_{H} \frac{\alpha(G \boxtimes H)}{\vartheta(G \boxtimes H)} = 1, \tag{1.26}$$

holds for every simple, finite, and undirected graph G, where the supremum is taken over all such graphs H.

Independence Numbers of Strong Powers of Graphs

Proposition: Let G be a finite, undirected, and simple graph. If $\alpha(\mathsf{G}^{\boxtimes \ell}) = \vartheta(\mathsf{G})^{\ell}$ for some $\ell \in \mathbb{N}$, then for every k that is an integral multiple of ℓ,

$$\alpha(\mathsf{G}^{\boxtimes k}) = \vartheta(\mathsf{G})^{k}. \tag{1.27}$$

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Proof

Let $k = \ell p$ with $p \in \mathbb{N}$. Then, since $\alpha(\mathsf{G} \boxtimes \mathsf{H}) \geq \alpha(\mathsf{G})\, \alpha(\mathsf{H})$ for all graphs G and H,

$$\vartheta(\mathsf{G})^{k} = \alpha(\mathsf{G}^{\boxtimes \ell})^{p} \leq \alpha(\mathsf{G}^{\boxtimes k}) \leq \vartheta(\mathsf{G}^{\boxtimes k}) = \vartheta(\mathsf{G})^{k}.$$

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Corollary 1: If $\alpha(\mathsf{G}) = \vartheta(\mathsf{G})$, then for all $k \in \mathbb{N}$, the k-fold strong power of G satisfies

$$\alpha(\mathsf{G}^{\boxtimes k}) = \vartheta(\mathsf{G})^{k}, \quad \forall\, k \in \mathbb{N}. \tag{1.28}$$

Example: the Tietze Graph (I.S., '23)

Let G be the Tietze graph, which is a 3-regular graph on 12 vertices that is not strongly regular, nor vertex- or edge-transitive.

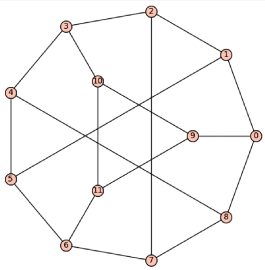

Figure 5: Tietze graph.

It can be verified that

$$\alpha(\mathsf{G}) = 5 = \vartheta(\mathsf{G}) \implies \alpha(\mathsf{G}^{\boxtimes k}) = 5^k, \ \forall\, k \in \mathbb{N}.$$

Example: the Tietze Graph (Cont.)

1. A largest independent set of G is $\{0, 3, 5, 7, 11\}$, so $\alpha(\mathsf{G}) = 5$.
2. The result $\vartheta(\mathsf{G}) = 5$ is obtained by solving the SDP problem:

$$\begin{array}{l} \text{maximize} \;\; \text{Trace}(\mathbf{B}\,\mathbf{J}_{12}) \\ \text{subject to} \\ \begin{cases} \mathbf{B} \in \mathcal{S}_+^{12}, \;\; \text{Trace}(\mathbf{B}) = 1, \\ A_{i,j} = 1 \; \Rightarrow \; B_{i,j} = 0, \quad i, j \in \{1, \ldots, 12\}. \end{cases} \end{array}$$

$$\Rightarrow \; B = \frac{1}{15} \begin{pmatrix} 2 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \; \vartheta(\mathsf{G}) = \text{Trace}(\mathbf{B}\,\mathbf{J}_{12}) = 5.$$

Example: the Tietze Graph (Cont.)

For comparison, since the Tietze graph is 3-regular on 12 vertices with $\lambda_{\min}(\mathsf{G}) = -2.30278$, the Hoffman-Lovász bound on $\vartheta(\mathsf{G})$ is equal to

$$\vartheta(\mathsf{G}) \leq -\frac{n\lambda_{\min}(\mathsf{G})}{d - \lambda_{\min}(\mathsf{G})} = 5.21110, \qquad (1.29)$$

so it is not tight. The fact that the bound is not tight is consistent with the fact that G is not an edge-transitive graph.

By Corollary 1 and our closed-form expression for the Lovász ϑ-function of SRGs, we calculate $\alpha(\mathsf{G}^{\boxtimes k})$ for some SRGs.

Independence Numbers of All Strong Powers of SRGs

1. The Hall-Janko graph G is srg$(100, 36, 14, 12)$, and $\alpha(\mathsf{G}^{\boxtimes k}) = 10^k$.
2. The Hoffman-Singleton graph G is srg$(50, 7, 0, 1)$, and $\alpha(\mathsf{G}^{\boxtimes k}) = 15^k$.
3. The Janko-Kharaghani graphs of orders 936 and 1800 are srg$(936, 375, 150, 150)$ and srg$(1800, 1029, 588, 588)$, respectively. For both graphs $\alpha(\mathsf{G}^{\boxtimes k}) = 36^k$.
4. Janko-Kharaghani-Tonchev: $\mathsf{G} =$ srg$(324, 153, 72, 72), \alpha(\mathsf{G}^{\boxtimes k}) = 18^k$.
5. The graphs introduced by Makhnev are $\mathsf{G} =$ srg$(64, 18, 2, 6)$ and $\overline{\mathsf{G}} =$ srg$(64, 45, 32, 30)$. We have $\alpha(\mathsf{G}^{\boxtimes k}) = 16^k$, and $\alpha(\overline{\mathsf{G}}^{\boxtimes k}) = 4^k$.
6. The Mathon-Rosa graph G is srg$(280, 117, 44, 52)$: $\alpha(\mathsf{G}^{\boxtimes k}) = 28^k$.
7. The Schläfli graph G is srg$(27, 16, 10, 8)$, and $\alpha(\mathsf{G}^{\boxtimes k}) = 3^k$.
8. The Shrikhande graph is srg$(16, 6, 2, 2)$; its capacity is $\alpha(\mathsf{G}^{\boxtimes k}) = 4^k$.
9. The Sims-Gewirtz graph G is srg$(56, 10, 0, 2)$, and $\alpha(\mathsf{G}^{\boxtimes k}) = 16^k$.
10. The graph G by Tonchev is srg$(220, 84, 38, 28)$, and $\alpha(\mathsf{G}^{\boxtimes k}) = 10^k$.

Theorem 1.8 (chromatic number of strong product of SRGs (I.S.,'23))

Let $\mathsf{G}_1, \ldots, \mathsf{G}_k$ be strongly regular graphs $\text{srg}(n_\ell, d_\ell, \lambda_\ell, \mu_\ell)$ for $\ell \in [k]$ (they need not be distinct). Then, the chromatic number of their strong product satisfies

$$\left\lceil \prod_{\ell=1}^{k} \left(1 + \frac{2d_\ell}{t_\ell + \mu_\ell - \lambda_\ell}\right) \right\rceil \leq \chi(\mathsf{G}_1 \boxtimes \ldots \boxtimes \mathsf{G}_k) \leq \prod_{\ell=1}^{k} \chi(\mathsf{G}_k), \tag{1.30}$$

where $\{t_\ell\}_{\ell=1}^k$ in the leftmost term is given by

$$t_\ell \triangleq \sqrt{(\lambda_\ell - \mu_\ell)^2 + 4(d_\ell - \mu_\ell)}, \quad \ell \in [k]. \tag{1.31}$$

The above lower bound is also larger than or equal to the product of the clique numbers of the factors $\{\mathsf{G}_\ell\}_{\ell=1}^k$.

Example: Chromatic Numbers of Strong Products

Let

$$\mathsf{G} \in \mathsf{srg}(27, 16, 10, 8), \quad \mathsf{H} \in \mathsf{srg}(16, 6, 2, 2), \quad \mathsf{J} \in \mathsf{srg}(100, 36, 14, 12)$$

be the Schläfli, Shrikhande, and Hall-Janko graphs, respectively.
The upper and lower bounds (in the previous slide) coincide here: for all integers $k_1, k_2, k_3 \geq 0$,

$$\chi(\mathsf{G}^{\boxtimes k_1} \boxtimes \mathsf{H}^{\boxtimes k_2} \boxtimes \mathsf{J}^{\boxtimes k_3}) = 9^{k_1}\, 4^{k_2}\, 10^{k_3}. \tag{1.32}$$

For comparison, the lower bound that is given by the product of the clique numbers of each factor is looser, and it is equal to $6^{k_1} 3^{k_2} 4^{k_3}$.

Shannon Capacity of a Graph

- A discrete channel consists of
  - a finite input set $\mathcal{X}$;
  - a (possibly infinite) output set $\mathcal{Y}$
  - a non-empty fan-out set $\mathcal{S}_x \subseteq \mathcal{Y}$ for every $x \in \mathcal{X}$.

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- In each channel use, a sender transmits an input $x \in \mathcal{X}$ and a receiver receives an arbitrary output in $\mathcal{S}_x$.

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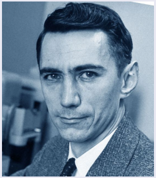
- In each channel use, a sender transmits an input $x \in \mathcal{X}$ and a receiver receives an arbitrary output in $\mathcal{S}_x$.
- Shannon (1956) initiated the study of the maximum amount (rate) of information that a channel can communicate without error.

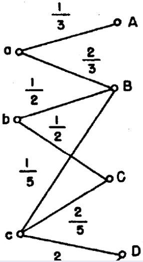


THE ZERO ERROR CAPACITY OF A NOISY CHANNEL

Claude E. Shannon
Bell Telephone Laboratories, Murray Hill, New Jersey
Massachusetts Institute of Technology, Cambridge, Mass.

Abstract

The zero error capacity C_o of a noisy channel is defined as the least upper bound of rates at which it is possible to transmit information with zero probability of error. Various properties of C_o are studied; upper and lower bounds and methods of evaluation of C_o are given. Inequalities are obtained for the C_o relating to the "sum" and "product" of two given channels. The analogous problem of zero error capacity C_{oF} for a channel with a feedback link is considered. It is shown that while the ordinary capacity of a memoryless channel with feedback is equal to that of the same channel without feedback, the zero error capacity may be greater. A solution is given to the problem of evaluating C_{oF}.



Shannon Capacity of a Graph (Cont.)

A discrete memoryless channel is represented by a *confusion graph* G:

- V(G) represent the symbols of the input alphabet to that channel.

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A discrete memoryless channel is represented by a *confusion graph* G:

- V(G) represent the symbols of the input alphabet to that channel.
- E(G): Two distinct vertices in G are adjacent if the corresponding two input symbols (say $x, x' \in \mathcal{X}$) are not distinguishable by the channel.

$$\mathsf{V}(\mathsf{G}) = \mathcal{X},$$
$$\mathsf{E}(\mathsf{G}) = \bigl\{\{x, x'\} : \, x, x' \in \mathcal{X}, \, x \neq x', \, \mathcal{S}_x \cap \mathcal{S}_{x'} \neq \varnothing\bigr\}.$$

(Both distinct input symbols can result in the same output.)

Shannon Capacity of a Graph (Cont.)

The largest number of inputs a channel can communicate without error in a single use is $\alpha(\mathsf{G})$ (the independence number of G).

Shannon Capacity of a Graph (Cont.)

The largest number of inputs a channel can communicate without error in a single use is $\alpha(\mathsf{G})$ (the independence number of G):

- The sender and the receiver agree in advance on an independent set $\mathcal{I}$ of a maximum size $\alpha(\mathsf{G})$.
- The sender transmits only inputs in $\mathcal{I}$.
- Every received output is in the fan-out set of exactly one input in $\mathcal{I}$.
  $\Rightarrow$ the receiver can correctly determine the transmitted input.

Shannon Capacity of a Graph (Cont.)

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$$y_i \in \mathcal{S}_{x_i}, \quad i = 1, \dots, k.$$

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- k uses of the channel are viewed as a single use of a larger channel:
  - its input set is $\mathcal{X}^k$, and its output set is $\mathcal{Y}^k$.
  - The fan-out set of $(x_1, \ldots, x_k) \in \mathcal{X}^k$ is the Cartesian product

$$\mathcal{S}_{x_1} \times \ldots \mathcal{S}_{x_k}.$$

Shannon Capacity of a Graph (Cont.)

- The *k-th confusion graph* is the k-fold strong power of G:

$$\mathsf{G}^{\boxtimes k} \triangleq \mathsf{G} \boxtimes \ldots \boxtimes \mathsf{G}. \tag{2.1}$$

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- $\alpha(\mathsf{G}^{\boxtimes k})$ is the max. number of k-length strings at the channel input that are distinguishable by the channel (error-free communication).
- The maximum *information rate per symbol* that is achievable by using input strings of length k is equal to

$$\frac{1}{k} \log \alpha(\mathsf{G}^{\boxtimes k}) = \log \sqrt[k]{\alpha(\mathsf{G}^{\boxtimes k})}, \quad k \in \mathbb{N}. \tag{2.2}$$

Shannon Capacity of a Graph (Cont.)

- The Shannon capacity of a graph G is defined to be the supremum of the maximum information rate over k (the length k of the input strings can be made as large as we wish):

$$\Theta(\mathsf{G}) = \sup_{k \in \mathbb{N}} \sqrt[k]{\alpha(\mathsf{G}^{\boxtimes k})} = \lim_{k \to \infty} \sqrt[k]{\alpha(\mathsf{G}^{\boxtimes k})}. \tag{2.3}$$

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The last equality holds by Fekete's Lemma:
the sequence $\{\alpha(\mathsf{G}^{\boxtimes k})\}_{k=1}^{\infty}$ is super-multiplicative, i.e.,

$$\alpha(\mathsf{G}^{\boxtimes (k_1+k_2)}) \geq \alpha(\mathsf{G}^{\boxtimes k_1}) \, \alpha(\mathsf{G}^{\boxtimes k_2}). \quad (2.4)$$

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Alas, the Shannon capacity can be rarely computed exactly !

On the Computability of the Shannon Capacity of Graphs

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  N. Alon and E. Lubetzky proved that the series of independence numbers in strong powers of a fixed graph can exhibit a complex structure, implying that the Shannon capacity of a graph cannot be approximated (up to a subpolynomial factor of the number of vertices) by any arbitrarily large, yet fixed, prefix of the series. This is true even if this prefix shows a significant increase of the independence number at a given power, after which it stabilizes for a while (IEEE T-IT, May 2006).

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- However, the Lovász ϑ-function of a graph is a computable (and sometimes tight) upper bound on the Shannon capacity. ☺

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On the Shannon Capacity of a Graph

LÁSZLÓ LOVÁSZ

Abstract—**It is proved that the Shannon zero-error capacity of the pentagon is $\sqrt{5}$. The method is then generalized to obtain upper bounds on the capacity of an arbitrary graph. A well-characterized, and in a sense easily computable, function is introduced which bounds the capacity from above and equals the capacity in a large number of cases. Several results are obtained on the capacity of special graphs; for example, the Petersen graph has capacity four and a self-complementary graph with n points and with a vertex-transitive automorphism group has capacity $\sqrt{n}$.**

A general upper bound on $\Theta(G)$ was also given in [6] (this bound was discussed in detail by Rosenfeld [5]). We assign nonnegative weights $w(x)$ to the vertices x of G such that

$$\sum_{x \in C} w(x) \leqslant 1$$

for every complete subgraph C in G; such an assignment is called a *fractional vertex packing*. The maximum of

Lovász Bound on the Shannon Capacity of Graphs (1979)

Theorem 2.1

For every finite, simple and undirected graph G,

$$\Theta(\mathsf{G}) \leq \vartheta(\mathsf{G}). \tag{2.5}$$

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Proof

$$\begin{aligned} \Theta(\mathsf{G}) &= \lim_{k\to\infty} \sqrt[k]{\alpha(\mathsf{G}^{\boxtimes\, k})} && (2.6)\\ &\leq \lim_{k\to\infty} \sqrt[k]{\vartheta(\mathsf{G}^{\boxtimes\, k})} && (2.7)\\ &= \vartheta(\mathsf{G}) && (2.8) \end{aligned}$$

where the last equality holds since $\vartheta(\mathsf{G}^{\boxtimes\, k}) = \vartheta(\mathsf{G})^k$ forall $k \in \mathbb{N}$.

In some cases, the capacity of a graph can be calculated exactly :)

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Theorem 2.2 (Lovász, 1979)

Let G be a self-complementary and vertex-transitive graph on n vertices. Then,

$$\Theta(\mathsf{G}) = \sqrt{n} = \vartheta(\mathsf{G}), \tag{2.9}$$

$$\alpha(\mathsf{G} \boxtimes \mathsf{G}) = n. \tag{2.10}$$

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Theorem 2.3 (Lovász, 1979)

Let $\mathsf{G} = \mathsf{K}(n, k)$ be a non-empty Kneser graph $(n \geq 2k)$. Then,

$$\alpha(\mathsf{G}) = \Theta(\mathsf{G}) = \vartheta(\mathsf{G}) = \binom{n-1}{k-1}. \tag{2.11}$$

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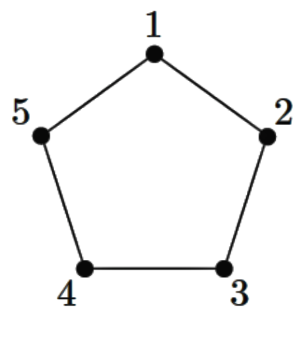
Theorem 2.4 (I.S., '24)

The same result in Theorem 2.2 for self-complementary vertex-transitive graphs also holds for self-complementary strongly regular graphs.

Example: Shannon Capacity of a 5-Cycle Graph (Lovász, 1979)

The pentagon (5-cycle) C_5 is self-complementary and vertex-transitive, so

$$\Theta(\mathsf{C}_5) = \sqrt{5} = \vartheta(\mathsf{C}_5), \quad \alpha(\mathsf{G} \boxtimes \mathsf{G}) = 5. \tag{2.12}$$



$\alpha(\mathsf{C}_5 \boxtimes \mathsf{C}_5) = 5$

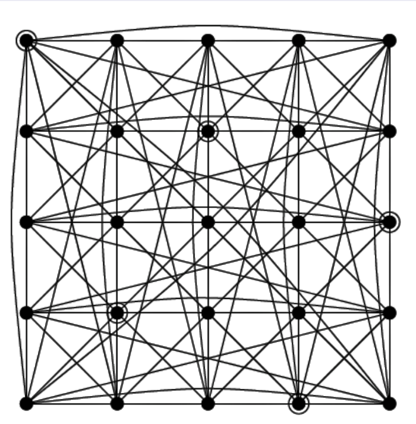
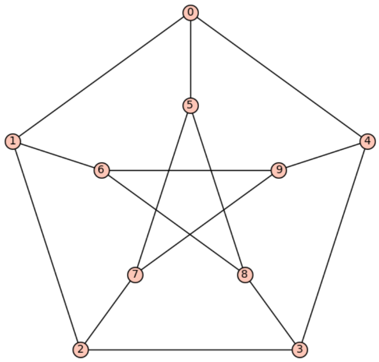


Figure 6: $\mathsf{C}_5 \boxtimes \mathsf{C}_5$. Independent set: $\{(1,1),\ (2,3),\ (3,5),\ (4,2),\ (5,4)\}$.

Petersen Graph

Example 2: Capacity of the Petersen Graph (Lovász, 1979)

The Petersen graph is isomorphic to $\mathsf{K}(5, 2)$, so its capacity is equal to 4.



Petersen Graph

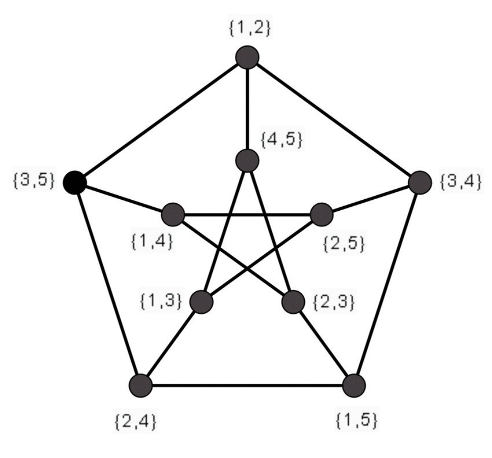


Figure 7: Petersen graph is isomorphic to the Kneser graph $\mathrm{K}(5, 2)$.

Shannon Capacities of Graphs: Recent Results (I.S.)

1. The Shannon capacity of two infinite subclasses of strongly regular graphs are determined in our paper (I.S. '24), as well as an extension of some prior results by Lovász (1979).

Shannon Capacities of Graphs: Recent Results (I.S.)

1. The Shannon capacity of two infinite subclasses of strongly regular graphs are determined in our paper (I.S. '24), as well as an extension of some prior results by Lovász (1979).
2. Our work (I.S., '24) also resolves a query regarding the variant of the ϑ-function by Schrijver and the identical function by McEliece *et al.* (1978). It shows, by a counterexample, that the ϑ-function variant by Schrijver does not possess the property of the Lovász ϑ-function of forming an upper bound on the Shannon capacity of a graph.

Recent Publications

This talk presents in part results from our recent journal papers:

1. I.S., "Observations on the Lovász ϑ-function, graph capacity, eigenvalues, and strong products," *Entropy*, vol. 25, no. 1, paper 104, pp. 1–40, January 2023. https://doi.org/10.3390/e25010104
2. I.S., "Observations on graph invariants with the Lovász ϑ-function," *AIMS Mathematics*, vol. 9, pp. 15385–15468, April 2024. https://doi.org/10.3934/math.2024747
3. I.S., "On strongly regular graphs and the friendship theorem," *Mathematics*, vol. 13, paper 970, pp. 1–21, March 2025. https://doi.org/10.3390/math13060970